Structure of the automorphism group of the augmented cube graph

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Abstract

The augmented cube graph AQ_n is the Cayley graph of $\mathbb{Z}_2^n$ with respect to the set of $2n-1$ generators $\{e_1, e_2, \ldots, e_n, 00\ldots0011, 00\ldots0111, 11\ldots1111\}$. It is known that the order of the automorphism group of the graph AQ_n is 2^{n+3}, for all $n \geq 4$. In the present paper, we obtain the structure of the automorphism group of AQ_n to be

$$\mathrm{Aut}(AQ_n) \cong \mathbb{Z}_2^n \rtimes D_8 \quad (n \geq 4),$$

where D_8 is the dihedral group of order 8. It is shown that the Cayley graph AQ_3 is non-normal and that AQ_n is normal for all $n \geq 4$. We also analyze the clique structure of AQ_4 and show that the automorphism group of AQ_4 is isomorphic to that of AQ_3:

$$\mathrm{Aut}(AQ_4) \cong \mathrm{Aut}(AQ_3) \cong (D_8 \times D_8) \rtimes C_2.$$

All the nontrivial blocks of AQ_4 are also determined.

Index terms — augmented cubes; automorphisms of graphs; normal Cayley graphs; clique structure; nontrivial block systems.

1. Introduction

Cayley graphs have been well-studied as a topology for interconnection networks due to their high symmetry, low diameter, high connectivity, and embeddable properties. Perhaps the most well-studied topology is the hypercube graph, for which there are many equivalent definitions. One definition is to view the hypercube as a Cayley graph.

Given a group H and a subset $S \subseteq H$, the Cayley graph of H with respect to S, denoted $\mathrm{Cay}(H, S)$, is the graph whose vertex set is H, and with arc set $\{(h, sh) :$

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$h \in H, s \in S\}$. If the identity group element e is not in H, then $\mathrm{Cay}(H,S)$ has no self-loops, and if H is closed under inverses, then (h,g) is an arc iff (g,h) is an arc; in this case, the Cayley graph can be viewed as an undirected graph. The Cayley graph $\mathrm{Cay}(H,S)$ is connected iff S generates H.

Let $\mathbb{Z}_2^n$ denote the direct product of n copies of the cyclic group of order 2. Let e_i denote the unit vector in the vector space $\mathbb{Z}_2^n$ that has a 1 in the ith coordinate and zero in the remaining coordinates. The hypercube graph is defined to be the Cayley graph of $\mathbb{Z}_2^n$ with respect to the set of generators $S = \{e_1, e_2, \ldots, e_n\}$.

The hypercube graph has been well-studied in the literature as a topology for interconnection networks. In order to improve its connectivity and fault-tolerance properties, the folded hypercube graph was proposed [8]. The folded hypercube graph is obtained by taking the hypercube graph and joining each vertex x to the unique vertex $\overline{x}$ that is farthest away from x in the hypercube. Thus, the folded hypercube is the Cayley graph of $\mathbb{Z}_2^n$ with respect to the set of generators $S = \{e_1, e_2, \ldots, e_n, 1\cdots 11\}$. The addition of these edges to the hypercube graph yields a new graph whose diameter is smaller.

The augmented cube graph was introduced in Choudum and Sunitha [2] as a topology for consideration in interconnection networks. The augmented cube graph of dimension n, denoted AQ_n, is the Cayley graph of $\mathbb{Z}_2^n$ with respect to the set of $2n-1$ generators $\{e_1, e_2, \ldots, e_n\} \mathbin{\dot\cup} \{00\cdots 0011,\ 00\cdots 0111, 00\cdots 1111, 11\cdots 1111\}$. Recall that the hypercube graph is the Cayley graph of $\mathbb{Z}_2^n$ with respect to the generator set $\{e_1, \ldots, e_n\}$. The additional $n-1$ generators in the definition of AQ_n augment the edge set of the hypercube graph, so that the augmented graph has smaller diameter and better connectivity and embeddable properties. For equivalent definitions of the augmented cube graph and a study of some of its properties, see [2] [3] [4].

Let $X = (V, E)$ be a simple, undirected graph. The automorphism group of X is defined to be the set of all permutations of the vertex set that preserves adjacency: $\mathrm{Aut}(X) := \{g \in \mathrm{Sym}(V) : E^g = E\}$. While one can often obtain some automorphisms of a graph, it is often difficult to prove that one has obtained the (full) automorphism group. Automorphism groups of graphs that have been proposed as the topology of interconnection networks have been investigated by several authors; see [5] [6] [10] [11] [9] [15] [18] [19].

Let $G := \mathrm{Aut}(AQ_n)$ denote the automorphism group of the augmented cube graph. Since $AQ_n = \mathrm{Cay}(\mathbb{Z}_2^n, S)$ is a Cayley graph, its automorphism group can be expressed as a rotary product $G = \mathbb{Z}_2^n \times_{rot} G_e$, where G_e is the stabilizer in G of the identity vertex e (cf. Jajcay [13] [14]). In [3] [4], it was shown that the order of the automorphism group of AQ_n is exactly 2^{n+3}, for all $n \geq 4$.

In the present paper, the structure of the automorphism group $G := \mathrm{Aut}(AQ_n)$ is determined to be $G \cong \mathbb{Z}_2^n \rtimes D_8$, for all $n \geq 4$. Hence, in addition to proving that $G_e \cong D_8$, we also strengthen the rotary product to a semidirect product. The clique structure of AQ_4 is investigated further, and we show that the automorphism group of AQ_4 is isomorphic to that of AQ_3: $\mathrm{Aut}(AQ_4) \cong \mathrm{Aut}(AQ_3) \cong (D_8 \times D_8) \rtimes C_2$. We also determine all the nontrivial blocks of AQ_4.

An open problem in the literature is to determine, given a group H and a subset $S \subseteq H$, whether $\mathrm{Cay}(H,S)$ is normal [17]. Every Cayley graph $\mathrm{Cay}(H,S)$ admits

$R(H)\,\mathrm{Aut}(H,S)$ as a subgroup of automorphisms; here $R(H)$ denotes the right regular representation of H, and $\mathrm{Aut}(H,S)$ is the set of automorphisms of H that fixes S setwise (cf. [1] [12]). A Cayley graph $\mathrm{Cay}(H,S)$ is said to be normal if its full automorphism group is $R(H)\,\mathrm{Aut}(H,S)$. In the present paper, we prove that the Cayley graph AQ_3 is non-normal, and we prove that AQ_n is normal for all $n \geq 4$.

Notation. We mention the notation used in this paper. Given a graph $X = (V,E)$ and a vertex $v \in V$, $X_i(v)$ denotes the set of vertices of X whose distance to vertex v is exactly i. Thus, if $X = \mathrm{Cay}(H,S)$, then $X_1(e) = S$. The subgraph of X induced by a subset of vertices $W \subseteq V$ is denoted $X[W]$, and so $X[X_i(v)]$ denotes the subgraph of X induced by the ith layer of the distance partition of X with respect to vertex v. The dihedral group of order $2n$ is denoted D_{2n}.

Given a Cayley graph $X = \mathrm{Cay}(H,S)$, e denotes the identity element of the group H and also denotes the corresponding vertex of X. Throughout this paper, H will be the abelian group $\mathbb{Z}_2^n$. In this case, $R(H) = R(\mathbb{Z}_2^n)$ is the set $\{\rho_z : z \in \mathbb{Z}_2^n\}$ of translations by z, i.e. $\rho_z : \mathbb{Z}_2^n \to \mathbb{Z}_2^n, x \mapsto x + z$, and ρ_z is an automorphism of the Cayley graph. We also let $\mathbb{Z}_2^n$ denote the vector space $\mathbb{F}_2^n$.

Let $G := \mathrm{Aut}(X)$. Then G_e is the set of automorphisms of X that fixes the vertex e, and L_e denotes the set of automorphisms of X that fixes the vertex e and each of its neighbors.

2. AQ_2 and AQ_3

Lemma 1. *Let AQ_2 be the augmented cube graph $\mathrm{Cay}(\mathbb{Z}_2^2, S)$, where $S = \{e_1, e_2, 11\}$. Then $\mathrm{Aut}(AQ_2) \cong S_4$ and AQ_2 is normal.*

Proof: Note that AQ_2 is the complete graph on 4 vertices. Hence, $\mathrm{Aut}(AQ_2) \cong S_4$ and the stabilizer in $G := \mathrm{Aut}(AQ_2)$ of the identity vertex e is $G_e \cong S_3$. Recall that the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^2, S)$ is normal iff $G_e \subseteq \mathrm{Aut}(\mathbb{Z}_2^n, S)$ (cf. Xu [17, Proposition 1.5]). The set $\mathrm{Aut}(\mathbb{Z}_2^3, S)$ of automorphisms of the group $\mathbb{Z}_2^2$ that fixes S setwise is exactly the set of invertible linear transformations of the vector space $\mathbb{Z}_2^2$ that fixes S setwise. The elements of $\mathrm{Aut}(\mathbb{Z}_2^2, S)$ are uniquely determined by their action on the basis $\{e_1, e_2\}$ of the vector space. Let $\phi \in \mathrm{Aut}(\mathbb{Z}_2^2, S)$. The image of $\phi(e_1)$ can be chosen from S in 3 ways, and the image $\phi(e_2)$ can be chosen from $S - \{\phi(e_1)\}$ in 2 ways. Hence, $|\mathrm{Aut}(\mathbb{Z}_2^2, S)| = 6$, which equals $|G_e|$. Thus, AQ_2 is normal, i.e. it has the smallest possible full automorphism group $R(\mathbb{Z}_2^2)\,\mathrm{Aut}(\mathbb{Z}_2^2, S) \cong \mathbb{Z}_2^2 \rtimes S_3$. ∎

Theorem 2. *Let AQ_3 be the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^3, S)$, where $S := \{e_1, e_2, e_3, 011, 111\}$. Let $G := \mathrm{Aut}(AQ_3)$. Then $G_e \cong D_8 \times C_2$, $\mathrm{Aut}(\mathbb{Z}_2^3, S) \cong D_8$, and AQ_3 is non-normal.*

Proof: Let $X := AQ_3$ and $G := \mathrm{Aut}(AQ_3)$. We show that the stabilizer in G of the vertex e is $G_e \cong D_8 \times C_2$ and that $\mathrm{Aut}(\mathbb{Z}_2^3, S) \cong D_8$, whence $\mathrm{Aut}(\mathbb{Z}_2^3, S) \neq G_e$ and AQ_3 is non-normal.

The distance partition of AQ_3 with respect to the identity vertex e is shown in Figure 1. The vertex e is adjacent to each vertex in $X_1(e)$ and each $u \in X_1(e) - \{011\}$

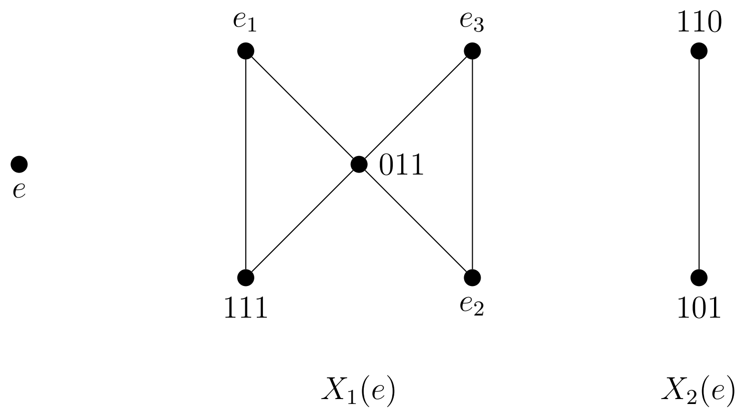

Figure 1: Distance partition of AQ_3 (edges between different layers are not drawn).

is adjacent to both vertices in $X_2(e)$ (these edges between the layers are not shown in the figure). The graph AQ_3 is 5-regular.

The subgraph of X induced by the ith layer of the distance partition is denoted $X[X_i(e)]$. The stabilizer G_e acts on the subgraph $X[X_i(e)]$ as a group of automorphisms. Observe from Figure 1 that an automorphism $g \in G_e$ can act on $X_1(e)$ by permuting the endpoints of the edge $\{e_1, 111\}$, by independently permuting the endpoints of the edge $\{e_2, e_3\}$, and by thereafter possibly also interchanging these two edges. Hence, the automorphism group of the induced subgraph $X[X_1(e)]$ is that of two independent edges (or its complement graph - the square) and hence is isomorphic to D_8.

The automorphism group of the induced subgraph $X[X_2(e)]$ is clearly C_2. Since each vertex in $X_2(e) - \{011\}$ is adjacent to all vertices in $X_1(e)$ and since G_e fixes the vertex 011, $G_e \cong D_8 \times C_2$. This also proves that $\mathrm{Aut}(AQ_3) = R(\mathbb{Z}_2^3) \times_{rot} (D_8 \times C_2)$.

We now determine $\mathrm{Aut}(\mathbb{Z}_2^3, S)$. Recall that the automorphisms of the group $\mathbb{Z}_2^n$ are precisely the invertible linear transformations of the vector space $\mathbb{Z}_2^n$ (cf. [7, p. 136]). Hence $\mathrm{Aut}(\mathbb{Z}_2^3, S)$ is precisely the set of elements of G_e that can be realized as linear transformations of the vector space $\mathbb{Z}_2^3$. We first show that $\mathrm{Aut}(X[X_1(e)]) \cong \mathrm{Aut}(\mathbb{Z}_2^3, S)$. Each automorphism ϕ of the induced subgraph $X[X_1(e)]$ also specifies the image $\phi(e_i)$ of the vertex e_i $(i = 1, 2, 3)$. The resulting image of the basis $\{e_1, e_2, e_3\}$ extends uniquely to a linear transformation ϕ' of the vector space $\mathbb{Z}_2^3$. Let $f : \mathrm{Aut}(X[X_1(e)]) \to \mathrm{Aut}(\mathbb{Z}_2^3, X_1(e)), \phi \mapsto \phi'$ be the map that takes an automorphism of the induced subgraph $X[X_1(e)]$ to its linear extension ϕ'. We show that the map f is well-defined and a bijective homomorphism.

Let $\phi \in \mathrm{Aut}(X[X_1(e)])$. Observe from Figure 1 that ϕ fixes the vertex 011 and permutes the vertices $X_1(e) - \{011\} = \{e_1, e_2, e_3, 111\}$ among themselves. In particular, if ϕ takes the basis element e_i to $\phi(e_i)$ $(i = 1, 2, 3)$, then ϕ must take the fourth vertex $111 = e_1 + e_2 + e_3$ to the remaining vertex (i.e. to the unique vertex in the singleton set $\{e_1, e_2, e_3, 111\} - \{\phi(e_1), \phi(e_2), \phi(e_3)\}$. This remaining vertex is the sum $\phi(e_1) + \phi(e_2) + \phi(e_3)$ because every 3 vectors in the set $\{e_1, e_2, e_3, 111\}$ sum to the fourth vector in the set. Also, the subsets $\{e_2, e_3\}$ and $\{e_1, 111\}$ are blocks of ϕ, and the sum of the two vertices in each subset is 011. In other words, if $\phi \in \mathrm{Aut}(X[X_1(e)])$,

then ϕ acts linearly on the vertices in $X_1(e)$, and so the extension of ϕ to a linear transformation ϕ' is well-defined.

An automorphism of the induced subgraph $X([X_1(e)])$ must take vertices e_1, e_2 and e_3 to 3 vertices in $\{e_1, e_2, e_3, 111\}$. Any 3 of these 4 vectors are linearly independent, whence the linear extension of ϕ is an invertible linear transformation. Thus, the map f is well-defined.

Different automorphisms of $X([X_1(e)])$ induce different images of the basis $\{e_1, e_2, e_3\}$ and hence different linear extensions. Thus, f is injective. Since $\mathrm{Aut}(\mathbb{Z}_2^3, S) \subseteq G_e$, every element $\phi' \in \mathrm{Aut}(\mathbb{Z}_2^3, S)$, when restricted to S, is an automorphism ϕ of $X([X_1(e)])$. The linear extension of ϕ is unique and hence equals ϕ'. Thus, f is surjective. The linear extension map is also a homomorphism. Hence f is an isomorphism.

Since $\mathrm{Aut}(X[X_1(e)]) \cong D_8$, we have that $\mathrm{Aut}(\mathbb{Z}_2^3, S) \cong D_8$. Thus, $\mathrm{Aut}(\mathbb{Z}_2^3, S)$ is a proper subgroup of $G_e = D_8 \times C_2$ and AQ_3 is non-normal. ∎

Lemma 3. *Let AQ_3 denote the augmented cube graph $\mathrm{Cay}(\mathbb{Z}_2^3, S)$ of dimension 3. Then $\mathrm{Aut}(AQ_3) \cong (D_8 \times D_8) \rtimes C_2$.*

Proof: Since $|S| = 5$, AQ_3 is a 5-regular graph on 8 vertices. Its complement graph $\overline{AQ_3}$ is $\mathrm{Cay}(\mathbb{Z}_2^3, S')$, where $S' := \mathbb{Z}_2^3 - S - \{0\} = \{110, 101\}$. It is easy to see that $\overline{AQ_3}$ is the union of two disjoint squares. Hence $\mathrm{Aut}(\overline{AQ_3}) \cong (D_8 \times D_8) \rtimes C_2$. ∎

3. Automorphism group and normality of AQ_n $(n \geq 4)$

Let AQ_n be the augmented cube graph $\mathrm{Cay}(\mathbb{Z}_2^n, S)$ $(n \geq 4)$. Let $G := \mathrm{Aut}(AQ_n)$. Since AQ_n is a Cayley graph, its automorphism group is the rotary product $R(\mathbb{Z}_2^n) \times_{rot} G_e$. It was shown in [4] [3] that the number of automorphisms of the graph AQ_n that fix the identity vertex e is exactly 8, i.e. $|G_e| = 8$. Thus, it was shown that the order of the automorphism group of AQ_n is $2^n \times 8 = 2^{n+3}$. In the present section, we determine the structure of the automorphism group of AQ_n: we show that $G_e \cong D_8$ and that every one of these 8 automorphisms can be realized as a linear transformation of the vector space $\mathbb{Z}_2^n$. It then follows that $G_e \subseteq \mathrm{Aut}(\mathbb{Z}_2^n, S)$ and that the rotary product expression can be strengthened to the semidirect product $G = R(\mathbb{Z}_2^n) \rtimes D_8$.

We first recall the following result from the literature:

Lemma 4. *[4] [3] Let $X := AQ_n = \mathrm{Cay}(\mathbb{Z}_2^n, S)$ be the augmented cube graph $(n \geq 4)$ and $G := \mathrm{Aut}(X)$. Then $|G_e| = 8$.*

We recall briefly the proof given in [4] for the result $|G_e| = 8$. First, it can be shown that the induced subgraph $X[X_1(e)]$ has exactly 8 automorphisms (cf. [4, Lemma 3.2]). It can also be shown that every automorphism of the graph AQ_n that fixes the identity vertex e and each of its neighbors is trivial, i.e. $L_e = 1$ (cf. [4, Theorem 3.1]). It follows that $|G_e| \leq 8$. Finally, Table 1 in [4] defines a set of 8 permutations of the vertex set of AQ_n that fix the identity vertex e and that are automorphisms

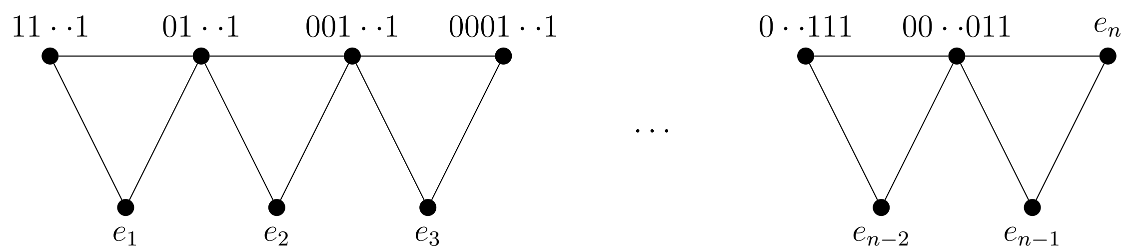

Figure 2: Induced subgraph of the first layer of the distance partition of AQ_n ($n \geq 4$).

of AQ_n. Hence $|G_e| = 8$. The paper [4] doesn't show the verification that the eight permutations given in [4, Table 1] preserve adjacency of AQ_n. In the present section, we express these 8 permutations as linear extensions of automorphisms of the induced subgraph $X[X_1(e)]$.

Proposition 5. *Let $X = AQ_n (n \geq 4)$ and $G := \mathrm{Aut}(X)$. Then each of the 8 elements in G_e can be realized as a linear transformation of the vector space $\mathbb{Z}_2^n$ that fixes S setwise, and so $G_e \subseteq \mathrm{Aut}(\mathbb{Z}_2^n, S)$.*

Proof: The induced subgraph $X[X_1(e)]$ is shown in Figure 2, where the subgraph is drawn horizontally. It can be seen from Figure 2 that automorphisms of the induced subgraph $X[X_1(e)]$ are realized by permuting the endpoints of the edge $\{e_1, 11\cdots1\}$, by independently permuting the endpoints of the edge $\{e_{n-1}, e_n\}$, and by possibly also interchanging these two edges (in which case the automorphism flips the graph left to right). Hence, $\mathrm{Aut}(X[X_1(e)]) \cong D_8$.

Define f_1 to be the map that takes vertex e_1 to $11\cdots1$, and e_i to $e_i (i = 2, \ldots, n)$. Let f_1' be the linear extension of f_1. Then f_1' takes $11\cdots1 = e_1 + \cdots + e_n$ to $f_1'(e_1) + \cdots + f_1'(e_n) = 11\cdots1 + e_2 + \cdots + e_n = e_1$. Similarly, f_1' takes $e_k + e_{k+1} + \cdots + e_n$ ($k \geq 2$) to itself. Thus, f_1' just interchanges the endpoints of the edge $\{e_1, 11\cdots1\}$ in the induced subgraph $X[X_1(e)]$. Since f_1' is an invertible linear transformation and fixes $X_1(e)$ setwise, $f_1' \in \mathrm{Aut}(\mathbb{Z}_2^n, S) \subseteq G_e$; hence, f_1' is an automorphism of the graph X. We have thus realized an element f_1' of G_e as a linear transformation of the vector space $\mathbb{Z}_2^n$.

Similarly, define f_2 to be the map that takes e_{n-1} to e_n, e_n to e_{n-1}, and e_i to e_i ($i = 1, 2, \ldots, n-2$). Define f_3 to be the map that interchanges e_1 and e_{n-1}, interchanges e_2 and e_{n-2}, and so on, and takes e_n to $11\cdots1$. Let f_2' and f_3' be the linear extensions of f_2 and f_3, respectively. It can be seen that the group of linear transformations $\langle f_1', f_2', f_3' \rangle$ is contained in G_e and is isomorphic to $\langle f_1, f_2, f_3 \rangle \cong D_8$. Thus, $G_e \cong D_8$ and $G_e \subseteq \mathrm{Aut}(\mathbb{Z}_2^n, S)$. ∎

We have proved above that

Theorem 6. *Let $X = AQ_n = \mathrm{Cay}(\mathbb{Z}_2^n, S) (n \geq 4)$ and $G := \mathrm{Aut}(X)$. Then $G_e \cong D_8$, AQ_n is a normal Cayley graph, and $\mathrm{Aut}(AQ_n) \cong \mathbb{Z}_2^n \rtimes D_8$.*

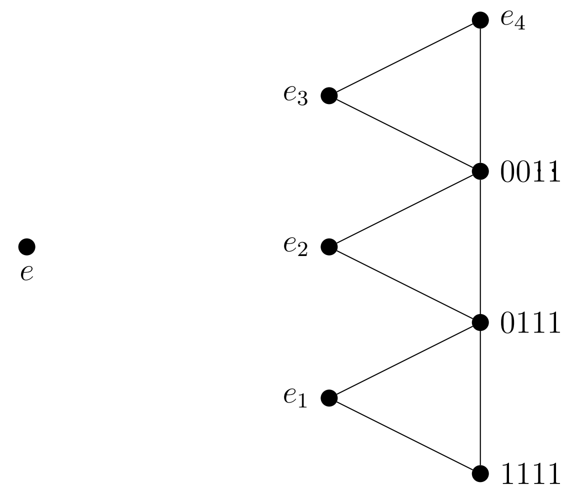

Figure 3: The identity vertex e and its neighbors in AQ_4.

4. Automorphism group of AQ_4

In this section, we study the clique structure of $X := AQ_4 = \mathrm{Cay}(\mathbb{Z}_2^4, S)$, where $S := \{e_1, e_2, e_3, e_4, 0011, 0111, 1111\}$. In particular, we describe the action of the automorphism group of AQ_4 on the set of all its maximum cliques. We show that the graph AQ_4 has 12 distinct maximum cliques and that the action of $G := \mathrm{Aut}(AQ_4)$ on a set of 8 maximum cliques is isomorphic to the action of the automorphism group of two disjoint squares acts on its vertices. It is then deduced that $\mathrm{Aut}(AQ_4) \cong (D_8 \times D_8) \rtimes C_2$.

Because X is vertex-transitive, there exists a maximum clique in the graph which contains the identity vertex e. Observe from Figure 3 that the size of a maximum clique in the induced subgraph $X[X_1(e)]$ is 3. Hence the clique number $\omega(X) = 4$.

There are exactly 3 4-cliques in AQ_4 containing the identity vertex e (see Figure 3). Hence, there are a total of $3 \times 16 = 48$ 4-cliques in AQ_4; however, in this count, each clique was counted 4 times, and so there are exactly 12 distinct maximum cliques in AQ_4, each of size 4.

Let $\mathcal{C}_1 := \{e, e_3, e_4, 0011\}$ be the set of 4 vertices of AQ_4 that form the maximum clique corresponding to the upper triangle in Figure 3. We call $\mathcal{C}_1$ an *upper clique* of AQ_4. Note that $\mathcal{C}_1$ is also a subspace of $\mathbb{Z}_2^4$ of dimension 2. Let $\mathcal{C}_2 := \mathcal{C}_1 + 1000 = \{e_1, 1010, 1001, 1011\}$ be the coset obtained by translating $\mathcal{C}_1$ by 1000. Let $\mathcal{C}_3 := \mathcal{C}_1 + 0100 = \{e_2, 0110, 0101, 0111\}$ and $\mathcal{C}_4 := \mathcal{C}_1 + 1100 = \{1100, 1110, 1101, 1111\}$. The set of upper cliques of AQ_4 is defined to equal $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$. The set of translations used to obtain these 4 4-cliques is $\{e, 1000, 0100, 1100\}$, a subspace of dimension 2 orthogonal to $\mathcal{C}_1$. Hence, the set of four upper cliques is a partition of the vertex set $\mathbb{Z}_2^4$ of AQ_4.

Similarly, a lower clique of AQ_4 is a 4-clique obtained by a translation of the clique $\mathcal{C}_5 := \{e, e_1, 0111, 1111\}$ corresponding to the lower triangle in Figure 3. The set of lower cliques is $\{\mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_7, \mathcal{C}_8\}$, where $\mathcal{C}_6 := \mathcal{C}_5 + e_4 = \{e_4, 1001, 0110, 1110\}$, $\mathcal{C}_7 := \mathcal{C}_5 + 0011 = \{0011, 1011, e_2, 1100\}$, and $\mathcal{C}_8 := \mathcal{C}_5 + e_3 = \{e_3, 1010, 0101, 1101\}$.

The set of middle cliques of AQ_4 of course consists of the clique $\mathcal{C}_9 := \{e, 0011, e_2, 0111\}$ corresponding to the middle triangle in Figure 3 and its translations $\mathcal{C}_{10} := \mathcal{C}_9 + e_4 = \{e_4, e_3, 0101, 0110\}$, $\mathcal{C}_{11} := \mathcal{C}_9 + e_1 = \{e_1, 1011, 1100, 1111\}$, and $\mathcal{C}_{12} := \mathcal{C}_9 + 1001 = \{1001, 1010, 1101, 1110\}$.

We have defined above the set $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{12}\}$ of all maximum cliques of AQ_4 in terms of translations (cosets) of 2-dimensional subspaces. This yields three different partitions of the vertex set of AQ_4, namely $\{\mathcal{C}_1, \ldots, \mathcal{C}_4\}$, $\{\mathcal{C}_5, \ldots, \mathcal{C}_8\}$, and $\{\mathcal{C}_9, \ldots, \mathcal{C}_{12}\}$. Only two of these three partitions are equivalent under the action of the automorphism group of AQ_4:

Lemma 7. *Let $G := \mathrm{Aut}(AQ_4)$, and let $\mathcal{C}_1, \ldots, \mathcal{C}_{12}$ be the 12 distinct maximum cliques of AQ_4 defined above. Then, the action of G on $\{\mathcal{C}_1, \ldots, \mathcal{C}_{12}\}$ has exactly 2 orbits: $\{\mathcal{C}_1, \ldots, \mathcal{C}_8\}$ and $\{\mathcal{C}_9, \ldots, \mathcal{C}_{12}\}$.*

Proof: Each of the upper cliques $\mathcal{C}_1, \ldots, \mathcal{C}_4$ is a translation of the upper clique $\mathcal{C}_1$. Because translations are automorphisms of the graph, all 4 upper cliques $\mathcal{C}_1, \ldots, \mathcal{C}_4$ lie in the same orbit of the action of G on maximum cliques. Similarly, the lower cliques $\mathcal{C}_5, \ldots, \mathcal{C}_8$ lie in the same G-orbit, and the middle cliques $\mathcal{C}_9, \ldots, \mathcal{C}_{12}$ lie in the same G-orbit.

The proof of Proposition 5 shows that there exists an element in G_e that takes the upper triangle (of Figure 3) to the lower triangle. Hence, the upper clique $\mathcal{C}_1$ and the lower clique $\mathcal{C}_5$ lie in the same G-orbit, whence $\{C_1, \ldots, C_8\}$ lies in a single G-orbit.

It now suffices to show that the middle clique $\mathcal{C}_9$ and upper clique $\mathcal{C}_1$ do not lie in the same G-orbit. The middle clique $\mathcal{C}_9 = \{e, 0011, e_2, 0111\}$ is a 2-dimensional subspace and is closed under addition. Hence, given any vertex $x \in \mathcal{C}_9$, there exists a translation in $\{\rho_z : z \in \mathcal{C}_9\}$ that takes x to the identity vertex e and that maps $\mathcal{C}_9$ to itself. Thus, if $\exists g \in G$ such that $g : \mathcal{C}_9 \mapsto \mathcal{C}_1$, then $\exists g \in G_e$ such that $g : \mathcal{C}_9 \mapsto \mathcal{C}_1$. But this is impossible since none of the 8 elements of $G_e \cong D_8$ take the middle triangle to the lower triangle (cf. Figure 3). Hence, the middle clique $\mathcal{C}_9$ and the upper clique $\mathcal{C}_1$ lie in different G-orbits. ∎

The argument above for AQ_4 can be generalized to show that the graph AQ_n ($n \geq 4$) has exactly $(n-1) \times \frac{2^n}{4} = (n-1)2^{n-2}$ distinct maximum cliques, and that the action of $\mathrm{Aut}(AQ_n)$ on this set of maximum cliques has exactly $\lfloor n/2 \rfloor$ orbits. In particular, it is easy to see that the 4-cliques corresponding to the first triangle and the last triangle of Figure 2 are in the same orbit, the 4-cliques corresponding to the second triangle and the last-but-one triangle are in the same orbit, and so on.

We showed in Lemma 7 that the set $\{\mathcal{C}_1, \ldots, \mathcal{C}_8\}$ of 8 cliques of AQ_4 forms a single orbit in the action of G on all maximum cliques. We study this restricted action of G (i.e. the transitive constituent) further:

Proposition 8. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_8$ be the maximum cliques of AQ_4 defined above. Then, the action of $G := \mathrm{Aut}(AQ_4)$ on $\{\mathcal{C}_1, \ldots, \mathcal{C}_8\}$ is faithful.*

Proof: Suppose $g \in G$ and g maps each $\mathcal{C}_i$ to itself ($i = 1, 2, \ldots, 8$). It suffices to show $g = 1$. Because g maps the lower clique $\mathcal{C}_5$ to itself and the upper clique $\mathcal{C}_1$ to itself, g maps their intersection $\{e\}$ to itself. Hence g fixes the vertex e. Similarly, g

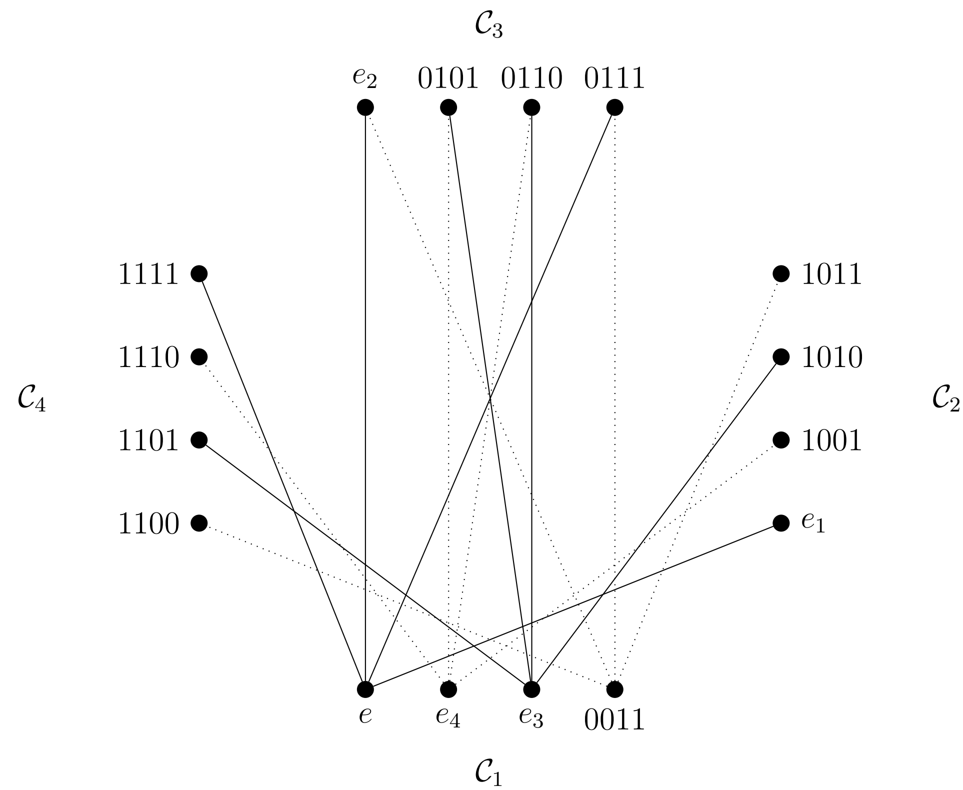

Figure 4: A partition of the vertex set of AQ_4 into cosets of clique $\mathcal{C}_1$.

fixes $\mathcal{C}_5 \cap \mathcal{C}_2 = \{e_1\}$ and g fixes $\mathcal{C}_1 \cap \mathcal{C}_8 = \{e_3\}$. But it is clear from Figure 3 that the only element from G_e that fixes e_1 and e_3 is the trivial automorphism, i.e. $g = 1$. ∎

If a group G acts on a set Ω, then a block of G is defined to be a subset $\Delta \subseteq \Omega$ such that $\Delta \cap \Delta^g = \Delta$ or $\Delta \cap \Delta^g = \phi$, for all $g \in G$.

Proposition 9. *Let $\mathcal{K} := \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$ and $\mathcal{K}' := \{\mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_7, \mathcal{C}_8\}$ denote the set of 4 upper cliques and the set of 4 lower cliques of AQ_4, respectively. Then $\mathcal{K}$ (and hence $\mathcal{K}'$) is a block of the action of G on $\{\mathcal{C}_1, \dots, \mathcal{C}_8\}$.*

Proof: Consider the partial drawing of AQ_4 shown in Figure 4. In this drawing, the 4 vertices in any upper clique are grouped together and the edges between vertices in the same upper clique are not drawn. Also, only all edges from the upper clique $\mathcal{C}_1 = \{e, e_3, e_4, 0011\}$ to vertices in the other three upper cliques are shown. The edges incident to consecutive vertices of $\mathcal{C}_1$ alternate between solid and dotted format, in order to distinguish between these edges more clearly. Note that the edges between $\mathcal{C}_1$ and $\mathcal{C}_3$ are twice as many as the edges between $\mathcal{C}_1$ and any other upper clique. Thus, $\{\mathcal{C}_1, \mathcal{C}_3\}$ is a block of G acting on $\mathcal{K}$. Studying the automorphism groups of graphs also helps to construct new drawings of graphs.

Observe from Figure 4 that the neighbors of $e \in \mathcal{C}_1$ in the three other upper cliques are $1111, e_2, 0111$ and e_1. Three of these neighbors, namely $e_1, 0111$ and 1111, along with e, form the lower clique $\mathcal{C}_5 = \{e, e_1, 0111, 1111\}$, whose 4 vertices lie in distinct

upper cliques. In other words, there exists a lower clique $\mathcal{C}_5$ that intersects all 4 upper cliques $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$.

Recall that G acts transitively on $\mathcal{K}$. Also, an automorphism of the graph must take pairwise disjoint cliques $\mathcal{C}_1, \ldots, \mathcal{C}_4$ to pairwise disjoint cliques. Hence, if $g \in G$ takes an upper clique, $\mathcal{C}_1$ say, to a lower clique $\mathcal{C}_5$, then g must take each upper clique to a lower clique, i.e. g must interchange $\mathcal{K}$ and $\mathcal{K}'$. Thus, each $g \in G$ either fixes $\mathcal{K} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$ setwise, or completely moves $\mathcal{K}$ to a set $\mathcal{K}'$ disjoint from $\mathcal{K}$, i.e. $\mathcal{K}$ is a block of the action of G on $\{\mathcal{C}_1, \ldots, \mathcal{C}_8\}$. ∎

In the next result, we essentially prove that the action of $G := \mathrm{Aut}(AQ_4)$ on the cliques $\mathcal{C}_1, \ldots, \mathcal{C}_8$ is isomorphic to the action of the automorphism group of two disjoint squares on the 8 vertices of the two squares.

Theorem 10. $\mathrm{Aut}(AQ_4) \cong (D_8 \times D_8) \rtimes C_2$.

Proof: By Lemma 7, $\{\mathcal{C}_1, \ldots, \mathcal{C}_8\}$ is an orbit of G acting on maximum cliques. Consider the action of G on $\{\mathcal{C}_1, \ldots, \mathcal{C}_8\}$. By Proposition 8, this induced action is faithful. Hence G is determined uniquely by the permutation group image of this induced action. By Proposition 9, $\mathcal{K} = \{\mathcal{C}_1, \ldots, \mathcal{C}_4\}$ and $\mathcal{K}' = \{\mathcal{C}_5, \ldots, \mathcal{C}_8\}$ are blocks of G.

Since $e_3 \in \mathcal{C}_1$, the translation $\rho_{e_3} \in G$ maps each upper clique $\mathcal{C}_i \in \mathcal{K}$ (which is a coset of the subspace $\mathcal{C}_1$) to itself. Hence ρ_{e_3} fixes $\mathcal{K}$ pointwise. On the other hand, ρ_{e_3} interchanges $\mathcal{C}_5$ and $\mathcal{C}_8$ because $e \in \mathcal{C}_5$ and $e_3 \in \mathcal{C}_8$. Also, ρ_{e_3} interchanges $\mathcal{C}_6$ and $\mathcal{C}_7$ because $e_4 \in \mathcal{C}_6$ and $e_4 + e_3 \in \mathcal{C}_7$. Thus, ρ_{e_3} effects the permutation $(\mathcal{C}_5, \mathcal{C}_8)(\mathcal{C}_6, \mathcal{C}_7) \in \mathrm{Sym}(\{\mathcal{C}_1, \ldots, \mathcal{C}_8\})$. Similarly, ρ_{e_4} fixes $\mathcal{K}$ pointwise and effects the permutation $(\mathcal{C}_5, \mathcal{C}_6)(\mathcal{C}_7, \mathcal{C}_8)$ on $\mathcal{K}'$.

Let A be the linear extension of the map $(e_1, e_2, e_3, e_4) \mapsto (e_1, e_2, e_4, e_3)$, i.e. A interchanges the last 2 coordinates of a vector. The coset $\mathcal{C}_1$ is closed under action by A and each vector in the span of $\{e_1, e_2\}$ is also fixed by A. So A maps the coset $\mathcal{C}_1 + a$ ($a \in \mathrm{Span}\{e_1, e_2\}$) to itself because $(\mathcal{C}_1 + a)^A = \mathcal{C}_1^A + a^A = \mathcal{C}_1 + a$. Thus, A fixes $\mathcal{K}$ pointwise.

The linear transformation A just defined is invertible. Note also that A fixes S setwise. Hence $A \in \mathrm{Aut}(\mathbb{Z}_2^4, S) \subseteq G_e$. Thus, A permutes the cliques $\mathcal{C}_1, \ldots, \mathcal{C}_8$ among themselves. Since A fixes $\mathcal{K}$ pointwise, A fixes $\mathcal{K}'$ setwise. It can be verified that A fixes $\mathcal{C}_5$ since $e \in \mathcal{C}_5$. Also, A interchanges $\mathcal{C}_6$ and $\mathcal{C}_8$ because $e_4 \in \mathcal{C}_6$ and $e_4^A = e_3 \in \mathcal{C}_8$. Thus, A effects the permutation $(\mathcal{C}_6, \mathcal{C}_8) \in \mathrm{Sym}(\{\mathcal{C}_1, \ldots, \mathcal{C}_8\})$.

Thus $N_1 := \langle \rho_{e_3}, \rho_{e_4}, A \rangle = \langle (\mathcal{C}_5, \mathcal{C}_8)(\mathcal{C}_6, \mathcal{C}_7), (\mathcal{C}_5, \mathcal{C}_6)(\mathcal{C}_7, \mathcal{C}_8), (\mathcal{C}_6, \mathcal{C}_8) \rangle \cong D_8$ (see Figure 5).

Similarly, consider $N_2 := \langle \rho_{e_1}, \rho_{1111}, A' \rangle$, where ρ_{e_1} maps each lower clique to itself and effects the permutation $(\mathcal{C}_1, \mathcal{C}_2)(\mathcal{C}_3, \mathcal{C}_4)$ on the upper cliques; ρ_{1111} effects the permutation $(\mathcal{C}_1, \mathcal{C}_4)(\mathcal{C}_2, \mathcal{C}_3)$. Let A' be the linear transformation that takes e_1 to 1111 and e_i to itself ($i = 2, 3, 4$). Then, A' fixes S setwise, whence $A' \in \mathrm{Aut}(\mathbb{Z}_2^4, S) \subseteq G_e$. It can be verified that A' effects the permutation $(\mathcal{C}_2, \mathcal{C}_4) \in \mathrm{Sym}(\{\mathcal{C}_1, \ldots, \mathcal{C}_8\})$. Thus, $N_2 \cong D_8$.

The permutations in N_1 fix each element of $\mathcal{K}$ and move only elements in $\mathcal{K}'$. The permutations in N_2 fix each element of $\mathcal{K}'$ and move only elements in $\mathcal{K}$. Hence, the elements of N_1 and N_2 commute with each other and $N_1 \cap N_2 = 1$. Thus, $N_1 N_2 = N_1 \times N_2 \cong D_8 \times D_8$.

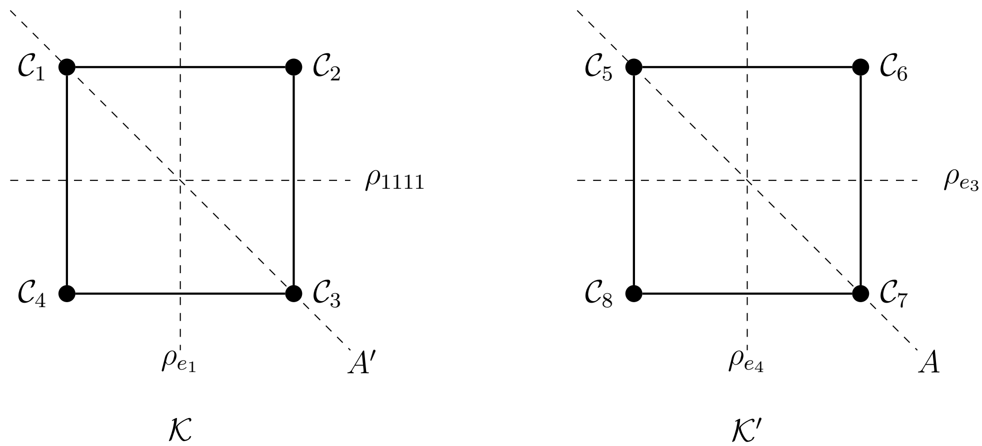

Figure 5: $G := \text{Aut}(AQ_4)$ acting on maximum cliques.

A permutation of $\mathcal{C}_1, \ldots, \mathcal{C}_8$ that interchanges $\mathcal{K}$ and $\mathcal{K}'$ is one induced by the linear map $B : (e_1, e_2, e_3, e_4) \mapsto (e_3, e_2, e_1, 1111)$ (cf. Figure 3). Then B interchanges the upper and lower cliques. The subgroup $\langle B \rangle \cong C_2$ normalizes $N_1 \times N_2$, which has index 2 in $(N_1 \times N_2)\langle B \rangle$. Thus, $(D_8 \times D_8) \rtimes C_2$ is isomorphic to a subgroup of G. By order considerations (cf. Theorem 6), $(D_8 \times D_8) \rtimes C_2$ is the full automorphism group G. ∎

5. Nontrivial blocks of AQ_4

In this section we obtain an equivalent condition for a subset of vertices of AQ_n to be a nontrivial block of the graph, and we determine all the nontrivial blocks of AQ_4. We first recall some definitions on blocks (cf. Wielandt [16], Biggs [1]).

Let $G \leq \text{Sym}(V)$ be a transitive permutation group. A block of G is a subset $\Delta \subseteq V$ that satisfies the condition $\Delta \cap \Delta^g = \Delta$ or $\Delta \cap \Delta^g = \phi$, for all $g \in G$. A block Δ of G is said to be nontrivial if $1 < |\Delta| < |V|$. The set $\{\Delta^g : g \in G\}$ of all translates of a block is called a complete block system (or a system of imprimitivity) and forms a partition of V. A permutation group G is said to be imprimitive if it has a nontrivial block.

A graph $X = (V, E)$ is imprimitive if it is transitive and its automorphism group is imprimitive. The blocks of X are defined to be the blocks of its automorphism group. Thus, if $W \subseteq V$ is a block of X and $g \in \text{Aut}(X)$, then W^g and W are either equal or disjoint (i.e. there cannot be overlap that is only partial). For example, for the hypercube graph, an antipodal pair $\{x, \overline{x}\}$ of vertices is a nontrivial block. The hypercube is bipartite, and the set of all even weight vertices is another nontrivial block. It can be shown (cf. Biggs [1, Proposition 22.3]) that the hypercube has only two nontrivial block systems.

We will use the following result:

Theorem 11. *[16, Theorem 7.4, 7.5] Let $G \leq \text{Sym}(V)$ be a transitive permutation group and let $e \in V$. Then, the lattice of blocks of G which contain e is isomorphic*

to the lattice of subgroups of G that contain G_e. More specifically, the correspondence between blocks Δ and subgroups H is as follows: $\Delta := e^H$ (the orbit of e under the action of H), and $H := G_{\{\Delta\}}$ (the set of elements of G that fixes Δ setwise).

To obtain the blocks of the augmented cube graph AQ_n, we determine the subgroups of $G := \mathrm{Aut}(AQ_n)$ that contain G_e and then use Theorem 11. For $z \in \mathbb{Z}_2^n$, let $(\rho_z : \mathbb{Z}_2^n \to \mathbb{Z}_2^n, x \mapsto x + z)$ denote translation by z. For a subset $K \subseteq \mathbb{Z}_2^n$, let $R(K) := \{\rho_z : z \in K\}$. We say that a subset $K \subseteq \mathbb{Z}_2^n$ is closed under action by G_e if $k^g \in K$, for all $k \in K, g \in G_e$.

Lemma 12. *Let $X := \mathrm{Cay}(\mathbb{Z}_2^n, S)$ be a normal Cayley graph (here, X need not be AQ_n), and let $G := \mathrm{Aut}(X) = R(\mathbb{Z}_2^n) \rtimes G_e$. Let K be a subset of vertices of the graph which forms a subspace of $\mathbb{Z}_2^n$ and which is closed under action by G_e. Then $R(K)G_e$ is a subgroup of G.*

Proof: The group G is finite, so it suffices to show that $R(K)G_e$ satisfies closure. We need to show that for all $z_1, z_2 \in K$ and $g_1, g_2 \in G_e$, there exist $z_3 \in K$ and $g_3 \in G_e$ such that $(\rho_{z_1} g_1)(\rho_{z_2} g_2) = \rho_{z_3} g_3$. This latter equation is equivalent to the condition $x^{(\rho_{z_1} g_1)(\rho_{z_2} g_2)} = x^{\rho_{z_3} g_3}$, for all $x \in V(X)$. The left hand side is $x^{\rho_{z_1} g_1 \rho_{z_2} g_2} = (x + z_1)^{g_1 \rho_{z_2} g_2} = (x^{g_1} + z_1^{g_1} + z_2)^{g_2} = x^{g_1 g_2} + z_1^{g_1 g_2} + z_2^{g_2}$. Here, we used the fact that g_i is a linear transformation, which is the case because for a normal Cayley graph $G_e \subseteq \mathrm{Aut}(\mathbb{Z}_2^n)$. The right hand side is $x^{\rho_{z_3} g_3} = x^{g_3} + z_3^{g_3}$. So we get that

$$x^{g_1 g_2} + z_1^{g_1 g_2} + z_2^{g_2} = x^{g_3} + z_3^{g_3}, \forall x \in V(X).$$

Take $g_3 = g_1 g_2$. Then z_3 must satisfy the equation $z_1^{g_1 g_2} + z_2^{g_2} = z_3^{g_1 g_2}$, or equivalently, the equation $z_1 + z_2^{g_1^{-1}} = z_3$. By hypothesis, K is closed under addition and under action by $g_1 \in G_e$. Hence, there exists a $z_3 \in K$ satisfying the equation. It follows that $R(K)G_e$ is a subgroup of G. ∎

We now show that the subgroups $R(K)G_e$ obtained from Lemma 12 are all the subgroups of G that contain G_e:

Proposition 13. *Let $X := \mathrm{Cay}(\mathbb{Z}_2^n, S)$ be a Cayley graph and let $G := \mathrm{Aut}(X)$. If $G_e \leq H \leq G$, then $H = R(K)G_e$ for some subspace K of $\mathbb{Z}_2^n$ that is closed under action by G_e.*

Proof: Suppose H satisfies $G_e \leq H \leq G$. Then H can be expressed as a disjoint union of left cosets of G_e in H, i.e. $H = h_1 G_e \cup h_2 G_e \cup \ldots \cup h_k G_e$, where k is the index of G_e in H. Note that $h_i \in H \leq G = R(\mathbb{Z}_2^n)G_e$, and so each h_i can be expressed as $h_i = \rho_{z_i} g_i$ for some $z_i \in \mathbb{Z}_2^n, g_i \in G_e$. Then, $h_i G_e = \rho_{z_i} g_i G_e = \rho_{z_i} G_e$. Thus, $H = R(K)G_e$, where $K := \{z_1, \ldots, z_k\}$. This proves that if H is a subgroup of G which contains G_e, then $H = R(K)G_e$ for some subset $K \subseteq \mathbb{Z}_2^n$.

To show K is a subspace, suppose $z_1, z_2 \in K$ (it is clear K is nonempty). We show $z_1 + z_2 \in K$. By closure in the subgroup $R(K)G_e$, $(\rho_{z_1} g_1)(\rho_{z_2} g_2) \in R(K)G_e$, for all $g_1, g_2 \in G_e$. In particular, taking $g_1 = g_2 = 1$, we get that $\rho_{z_1}\rho_{z_2} \in R(K)G_e$. This implies that there exist $z_3 \in K$ and $g_3 \in G_e$ such that $\rho_{z_1+z_2} = \rho_{z_3} g_3$. The right hand side takes vertex z_3 to $z_3^{\rho_{z_3} g} = e^g = e$, whence $z_3^{\rho_{z_1+z_2}} = e$. But this implies that

$z_3 = z_1 + z_2$. Thus, for closure to hold in $R(K)G_e$, $R(K)$ must also contain $\rho_{z_1+z_2}$, i.e. K is closed under addition and hence is a subspace.

It remains to show that K is closed under action by G_e. Let $z_1 \in K, g_1 \in G_e$. We show $z_1^{g_1} \in K$. Since $R(K)G_e$ is a subgroup, there exist $z_2 \in K$ and $g_2 \in G_e$ such that $\rho_{z_1} g_1 = g_2 \rho_{z_2}$, i.e. $x^{\rho_{z_1} g_1 \rho_{z_2}} = x^{g_2}$, for all $x \in V(X)$. Taking $x = e$, we get $z_1^{g_1} = z_2$, i.e. $z_1^{g_1}$ is also in K. ∎

Thus, the subgroups H of G that contain G_e are in bijective correspondence with the subspaces K of $\mathbb{Z}_2^n$ that are closed under action by G_e. The conditions of Lemma 12 require K to be closed under action by G_e. This implies K is a union of orbits of G_e.

Suppose we have a subset $K \subseteq \mathbb{Z}_2^n$ satisfying the conditions of Lemma 12. By Theorem 11, the subgroup $H := R(K)G_e$ gives rise to the block $\Delta = e^H = e^{R(K)G_e} = K^{G_e} = K$. Given such a block Δ, a complete block system $\{\Delta^g : g \in G\}$ is the set of all translations of Δ: if $\rho_z g \in G$ ($z \in \mathbb{Z}_2^n, g \in G_e$), then $\Delta^{\rho_z g} = K^{\rho_z g} = (K + z)^g = K^g + z^g = K + z^g = \Delta + z^g$. Thus, to determine all nontrivial block systems of a normal Cayley graph $X = \mathrm{Cay}(\mathbb{Z}_2^n, S)$, it suffices to determine all subspaces K of $\mathbb{Z}_2^n$ that are closed under action by G_e.

By the arguments above, we get the following characterization for the nontrivial blocks of a normal Cayley graph:

Theorem 14. *Let $X := \mathrm{Cay}(\mathbb{Z}_2^n, S)$ be a normal Cayley graph and let $G := \mathrm{Aut}(X)$. Then, the nontrivial blocks of X that contain the identity vertex e are precisely the subspaces of $\mathbb{Z}_2^n$ that are closed under action by G_e.*

We now apply the above results to the special case where the graph is AQ_4.

Proposition 15. *Each of the following subsets is a nontrivial block of AQ_4: $\Delta = \{e, e_2\}$; $\Delta' := \{e, e_2, 0011, 0111\}$; $\Delta'' := \{e, e_2, 0011, 0111, 1001, 1010, 1101, 1110\}$.*

Proof: The vertex e_2 is fixed by G_e (cf. Figure 3). Hence, the subspace spanned by $\{e_2\}$ is fixed by G_e. By Theorem 14, this subspace is a block of AQ_4. The four vectors in Δ' form a 2-dimension subspace $\mathrm{Span}\{e_2, 0011\}$ of $\mathbb{Z}_2^4$ and this subset is closed under action by G_e (see Figure 3). By Theorem 14, Δ' is a block of AQ_4.

It suffices to show that Δ'' is a union of orbits of G_e and that Δ'' forms a subspace. It is clear from the drawing of AQ_4 (see Figure 3) that $\{e\}$, $\{e_2\}$ and $\{0011, 0111\}$ are fixed blocks (hence union of orbits) of G_e. Since G_e fixes e_2, G_e fixes the set $\{1001, 1010, 1101, 1110\}$ of non-neighbors of e_2 in $X_2(e)$ setwise. Thus, this set is a fixed block of G_e and Δ'' is a union of orbits of G_e. It can be verified that Δ'' is a subspace of dimension 3 spanned by $\{0011, 0100, 1001\}$. Hence $\Delta'' \subseteq \mathbb{Z}_2^4$ is such that $R(\Delta'')G_e$ is a subgroup, and hence, the corresponding set Δ'' is a block of G. ∎

Theorem 16. *The 3 nontrivial blocks given in Proposition 15 are the only nontrivial blocks of AQ_4 which contain the vertex e.*

Proof: By Theorem 14, it suffices to find all the subspaces K of $\mathbb{Z}_2^4$ that are closed under action by G_e. We can find the orbits of G_e, and determine which fixed blocks

(union of orbits) of G_e are subspaces. Since K must be nontrivial, $K \neq \{e\}$ and $K \neq \mathbb{Z}_2^4$.

It is clear from Figure 3 that $\Delta_1 := \{e\}$, $\Delta_2 := \{e_2\}$, $\Delta_3 := \{0011, 0111\}$ and $\Delta_4 := \{e_1, 1111, e_3, e_4\}$ are orbits of G_e. Note that Δ_4 spans $\mathbb{Z}_2^4$, and so K does not contain Δ_4. Define the linear transformations $A_1 : (e_1, e_2, e_3, e_4) :\mapsto (e_1, e_2, e_4, e_3)$, $A_2 : (e_1, e_2, e_3, e_4) \mapsto (1111, e_2, e_3, e_4)$, and $A_3 : (e_1, e_2, e_3, e_4) \mapsto (e_3, e_2, e_1, 1111)$. Then $\langle A_1, A_2, A_3 \rangle = G_e \cong D_8$.

Let $x := 0101$. Then $x^{A_1} = 0110$, $x^{A_3} = 1011$ and $x^{A_1 A_3} = 1100$. Hence, $\Delta_5 := \{0101, 0110, 1011, 1100\}$ lies in a single G_e-orbit. Let $y := 1110$. Then $y^{A_1} = 1101$, $y^{A_2} = 1001$ and $y^{A_1 A_2} = 1010$. Hence, $\Delta_6 := \{1110, 1101, 1001, 1010\}$ lies in a single G_e-orbit. If $\Delta_5 \dot\cup \Delta_6 \subseteq K$, then $|K| \geq |\Delta_5| + |\Delta_6| + |\{e\}| = 9$, but since K is a subspace, $|K|$ is a power of 2, and so $K = \mathbb{Z}_2^4$, a contradiction. Thus, at most one of Δ_5 or Δ_6 is contained in K. If $K \supseteq \Delta_5$, then $|K| \geq 5$, and so $|K| = 8$. But it can be verified that $\Delta_1 \dot\cup \Delta_2 \dot\cup \Delta_3 \dot\cup \Delta_5$ is not a subspace. The subset $\Delta_1 \dot\cup \Delta_2 \dot\cup \Delta_3 \dot\cup \Delta_6$ is the subspace Δ'' of Proposition 15. If K contains neither Δ_5 nor Δ_6, then the only choices for K are $\Delta_1 \dot\cup \Delta_2$ and $\Delta_1 \dot\cup \Delta_2 \dot\cup \Delta_3$ (denoted Δ and Δ', respectively, in Proposition 15). ∎

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